PROPAGATION OF ELASTIC STRESS WAVES IN CONTINUOUSLY INHOMOGENEOUS ANISOTROPIC MEDIA

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UDC 539.3+539.4

Anisotropic materials of fibrous structure are being more and more widely used in technology. A characteristic feature of these materials is the broad possibility of controlling structure. In view of this, structures and bodies made of them generally have not only anisotropic, but also continuously inhomogeneous properties. This fact together with the possibility of varying the character of the anisotropy and inhomogeneity necessitate a broad and detailed study of such bodies, including their behavior under dynamic loading. In the present article we investigate certain characteristics of the propagation of elastic waves in anisotropic and inhomogeneous media, in particular the possibility of dynamic compactness of stress waves.

1. Statement of the Problem. We consider a continuously inhomogeneous and anisotropic medium bounded by a surface S and extending beyond it to infinity. Up to the time $t = 0$ the medium is at rest. At $t = 0$ points on the surface S are disturbed by some system of loads which subsequently depend on time. The problem is to explain (in the linear formulation) some characteristics of the propagation of stress waves produced by these disturbances.

We write down the necessary relations [1]:

the equations of motion

$$\rho^{-1}\nabla_i \sigma^{ij} = \ddot{u}^j; \tag{1.1}$$

Hooke's law

$$\sigma^{ij} = C^{ijkl}\varepsilon_{kl}; \tag{1.2}$$

the Cauchy equations

$$\varepsilon_{kl} = \frac{1}{2}(\nabla_k u_l + \nabla_l u_k). \tag{1.3}$$

By using the symmetry properties of the stiffness tensor C^{ijkl} we obtain from (1.2) and (1.3)

$$\sigma^{ij} = C^{ijkl}\nabla_k u_l. \tag{1.4}$$

We form the following combination

$$C^k_{lmj}\nabla_k(\rho^{-1}\nabla_i\sigma^{ij} - \ddot{u}^j) = 0.$$

Using (1.4) we have

$$C^k_{lmj}\nabla_k(\rho^{-1}\nabla_i\sigma^{ij}) = \ddot{\sigma}_{lm}. \tag{1.5}$$

System (1.5) is the basic resolving system of the elasticity-theory stress equations in curvilinear coordinates for an inhomogeneous anisotropic body. Since the stress tensor is symmetric, and the components of the stiffness tensor C^k_{lmj} are symmetric in the indices l and m, there are six independent equations in the six unknowns σ_{lm}.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 5, pp. 126–134, September–October, 1979. Original article submitted October 23, 1978.

0021-8944/79/2005-0629$07.50

We set the following boundary conditions for system (1.5):

$$\sigma^{ij}\nu_j|_S = T^iH(t); \tag{1.6}$$

$$\sigma^{ij} = \dot{\sigma}^{ij} = 0 \quad \text{at} \quad t = 0; \tag{1.7}$$

$$\sigma^{ij} \to 0 \quad \text{as} \quad |x| \to \infty, \tag{1.8}$$

where H(t) is the Heaviside function; $|x|$, distance from the surface S; and ν_j, a unit vector normal to the surface.

Equations of type (1.5) were derived in cartesian coordinates for a homogeneous isotropic medium in a different way by Ignachak (cf. [2]) and written in the form

$$\frac{\rho}{\mu}\left(\ddot{\sigma}_{ij} - \frac{\lambda}{3\lambda + 2\mu}\delta_{ij}\ddot{\sigma}_{kk}\right) = \sigma_{ki,kj} + \sigma_{jk,ki}. \tag{1.9}$$

Setting

$$C_{lmjk} = \lambda\delta_{lm}\delta_{jk} + \mu(\delta_{lk}\delta_{mj} + \delta_{lj}\delta_{mk}), \ \rho = \text{const}$$

in (1.5), where λ and μ are the Lamé parameters, shows the equivalence of Eqs. (1.5) and (1.9).

2. Deviation of Resolvents of the Boundary-Value Problem. We assume that the mechanical parameters of the medium are such that at each fixed point in space the equations of motion (1.5) are hyperbolic. Therefore, the velocity of propagation of a disturbance must be finite, and in the solution of boundary-value problem (1.5)-(1.8) there must be a surface of discontinuity $\Omega(x_\alpha, t) = 0$ separating the disturbed region from the region at rest. Consequently, we seek the solution in the class of discontinuous functions.

We write the solution of the boundary-value problem as the product of a smooth function F^{ij} times the generalized Heaviside function H:

$$\sigma^{ij} = F^{ij}(x_\alpha,t)H(\Omega), \tag{2.1}$$

which automatically satisfies the radiation conditions (1.8). The surface of discontinuity (wavefront) is not known beforehand. It can be found from the equation determining the condition necessary for the existence of a solution of form (2.1).

We see that Ω is expressible in terms of t:

$$\Omega = t - \omega(x_\alpha), \tag{2.2}$$

where ω is a smooth function of coordinates and independent of time. Then (2.1) can be written as

$$\sigma^{ij} = F^{ij}(x_\alpha, t - \omega)H(t - \omega). \tag{2.3}$$

The form of Eq. (2.3) enables us to take the Laplace transform

$$\tilde{\sigma}^{ij} = \int_0^\infty \sigma^{ij}e^{-pt}dt. \tag{2.4}$$

Since the method of Laplace transformations (2.4) can operate with impulse functions, we assume their presence in our solution:

$$F^{ij} = z^{ij(-1)}\delta(t - \omega) + F_0^{ij}(x_\alpha, t - \omega), \tag{2.5}$$

where F_0^{ij} is the analytic part of the function F^{ij}. We expand F_0^{ij} in (2.5) in a Taylor series in time in the neighborhood of $t = \omega$, i.e., in the neighborhood of the wavefront. Then

$$F^{ij} = z^{ij(-1)}(x_\alpha)\,\delta(t - \omega) + \sum_{n=0}^{\infty}\frac{z^{ij(n)}(x_\alpha)}{n!}(t - \omega)^n. \tag{2.6}$$

Actually we have obtained the representation of the solution of (2.3) in the form of a ray expansion [3].

If we now take the Laplace transform of (2.1) and use (2.6) we obtain

$$\tilde{\sigma}^{ij} = \tilde{F}^{ij} e^{-p\omega} = \sum_{n=-1}^{\infty} \frac{z^{ij(n)}}{p^{n+1}} e^{-p\omega}. \quad (2.7)$$

The representation (2.6) and (2.7) enables us to take inverse transforms automatically.

A representation similar to (2.6) was used in [4] in a one-dimensional problem of a stress wave in an inhomogeneous rod to determine the change of the wave amplitude at the front during its propagation.

Using (1.7) we write the Laplace transform of system (1.5)

$$C^k_{lmj} \nabla_k \left(\rho^{-1} \nabla_i \tilde{\sigma}^{ij}\right) = p^2 \tilde{\sigma}_{lm}. \quad (2.8)$$

Substituting Eqs. (2.7) and (2.8), collecting terms in the same powers of p and equating them to zero, we obtain

$$-L_{lm}(z^{ij(n-2)}) + M_{lm}(z^{ij(n-1)}) = D_{lm}(z^{ij(n)}), \quad (2.9)$$

where

$$D_{lm}(z^{ij}) = \rho^{-1} C^k_{lmj} \omega_{,i} \omega_{,k} z^{ij} - z_{lm}; \quad (2.10)$$

$$M_{lm}(z^{ij}) = C^k_{lmj} \{\nabla_k (\rho^{-1} \omega_{,i} z^{ij}) + \rho^{-1} \omega_{,k} \nabla_i z^{ij}\}; \quad (2.11)$$

$$L_{lm}(z^{ij}) = C^k_{lmj} \nabla_k (\rho^{-1} \nabla_i z^{ij}) \quad (2.12)$$

$(n = -1, 0, 1, 2, \ldots, i, j = 1, 2, 3)$.

Here

$$z^{ij(-2)} \equiv z^{ij(-3)} \equiv 0.$$

3. Solvability of the System of Recurrence Equations (2.9). It can be shown that for (2.9) to be solvable it is necessary that the system of algebraic equations

$$\left(\rho^{-1} C^k_{lmj} \omega_{,i} \omega_{,k} - g_{lj} g_{mi}\right) z^{ij(-1)} = 0 \quad (3.1)$$

have a nontrivial solution. The necessary and sufficient condition for this is the vanishing of the determinant of the matrix of the unknowns [5]. If we write the matrix in (3.1) in cartesian coordinates, a linear transformation of its rows and columns (preserving the determinant) can be found which reduces it to the form

$$\begin{Vmatrix} C^{jk}_{il}\omega_{,k}\omega_{,l} - \rho\delta^j_i & . & 0 & 0 & 0 \\ & . & 0 & 0 & 0 \\ \ldots & . & 0 & 0 & 0 \\ C^{1k}_{12}\omega_{,k}; & C^{2k}_{12}\omega_{,k}; & C^{3k}_{12}\omega_{,k}; & -\rho;\ 0; & 0 \\ C^{1k}_{13}\omega_{,k}; & C^{2k}_{13}\omega_{,k}; & C^{3k}_{13}\omega_{,k}; & 0;\ -\rho; & 0 \\ C^{1k}_{23},\ \omega_{,k}; & C^{2k}_{23}\omega_{,k}; & C^{3k}_{23}\omega_{,k}; & 0;\ \ 0; & -\rho \end{Vmatrix} \quad (3.2)$$

(k and l are summed over from 1 to 3). Hence it is clear that

$$-\rho^3 \det \left| C^{jk}_{il} \omega_{,k} \omega_{,l} - \rho \delta^j_i \right| = 0. \quad (3.3)$$

The last equation is the characteristic equation for (1.5), whose surfaces are determined by the equation $\Omega = 0$ [6]. The boundary conditions for the function ω follow from the condition that at $t = 0$ the surface S must coincide with the wavefront surface, i.e.,

$$\omega |_S = 0. \quad (3.4)$$

If we introduce the idea of the velocity of the front along its normal, and the direction cosines of the normal, and use the equations [7]

$$G_n = -\dot{\Omega}|\text{grad } \omega|^{-1} = |\text{grad } \omega|^{-1}, \quad \nu_j = -\Omega_{,j}|\text{grad } \omega|^{-1} = \omega_{,j}G_n, \tag{3.5}$$

we can rewrite (3.3) in the equivalent form

$$\det|C_{il}^{jk}\nu_k\nu_l - \rho G_n^2\delta_i^j| = 0. \tag{3.6}$$

Thus, the problem of finding G_n^2 from (3.6) is reduced to the problem of determining the eigenvalues of the symmetric matrix $G_{il}^{jk}\nu_k\nu_l$. It is known that in this case all the eigenvalues of the matrix are real. Hence, it follows that along any given direction ν_j in an elastic medium there are exactly three possible velocities of wave propagation [8]. The problem of determining the function ω is described in somewhat more detail in [3].

It is known [5] that the application of elementary transformations with columns of a matrix is equivalent to a linear transformation of variables in (3.1). It turns out that for cartesian coordinates the transformation from matrix (3.1) to (3.2) is equivalent to introducing the new variables

$$y_i^{(-1)} = \sum_{j=1}^{3} z_{ij}^{(-1)}\omega_{,j} \quad (i = 1, 2, 3), \quad y_4^{(-1)} = z_{12}^{(-1)}, \quad y_5^{(-1)} = z_{13}^{(-1)}, \quad y_6^{(-1)} = z_{23}^{(-1)}. \tag{3.7}$$

By analogy with (3.7) where $n = -1$ we introduce variables $y_i^{(n)}$ for $n > -1$.

We transform the infinite system of equations obtained into groups of six equations each, which together with boundary conditions (1.6) permit the determination of $y_i^{(-1)}, \ldots, y_i^{(n)}, \ldots$, in succession, and consequently the components of the stress tensor (for a known function of the front ω). We assume that all values of ω satisfying Eqs. (3.3) and (3.4) have been determined, and that the functions are single-valued and sufficiently smooth functions of coordinates, thus excluding from consideration the presence of caustics.

We substitute one of these values into the transformed system of equations (2.9) for $n = -1$. From the form of matrix (3.2) it follows that six of the components $y_i^{(-1)}$ are expressed solely in terms of $r < 3$ functions of $y_1^{(-1)}$, $y_2^{(-1)}$, $y_3^{(-1)}$, where $3 - r$ is the rank of the corner minor in (3.2). In addition let us consider the first three of the transformed equations (2.9) for $n = 0$. We have a system of linear algebraic equations for $y_1^{(0)}$, $y_2^{(0)}$, $y_3^{(0)}$ with an inhomogeneous right-hand side depending on the given coefficients in the equations, the known solution for ω, and $y_i^{(-1)}$. It is known [5] that for such a system of equations there are exactly r linearly independent transformations of rows giving a zero combination on the left-hand side of the system. This enables us to obtain r first-order partial differential equations for the r unknown functions of the $y_i^{(-1)}$. We go through a similar process for the remaining solutions for ω. As a result we arrive at the problem of determining r_1 functions from the same number of columns of the equations, where r_1 is the total number of functions to be determined

The rank of the corner minor of (3.2) can depend on both the point in space x_α and the choice of the coordinate system (the direction of the gradient of ω). Using the representation (3.6) and reducing this matrix to the Jordan form, it can be shown that if at a given point for given directions its rank is $3 - r$, the multiplicity of the velocity G_n in this case is r. It follows from this that for any coordinates at each point in space finding the $y_i^{(-1)}$ $(i = 1, 2, 3)$ is reduced to the solving of a system of three first-order partial differential equations. The general solution of the problem is written as a sum of particular solutions, and must satisfy the boundary conditions on the surface S. Let us obtain these conditions. We expand the transform of the right-hand side of (1.6) in the series

$$\tilde{T}^i = \sum_{n=k}^{\infty} \frac{T^{i(n)}}{p^{n+1}} \quad (k \geqslant -1). \tag{3.8}$$

We rewrite (1.6) using Eqs. (3.5) and (3.7)

$$\sigma^{ij}\omega_{,j}|_S = y^i|_S = G_n^{-1}|_S T^i H(t).$$

Hence

$$y^{i(n)}|_S = G_n^{-1}|_S T^{i(n)} \quad (i = 1, 2, 3).$$

Consequently the problem of determining the $y_i^{(-1)}$ ($i = 1, 2, \ldots, 6$) is reduced to a Cauchy boundary-value problem. Henceforth, we assume that the boundary S, the coefficients in Eqs. (1.5), and the boundary functions T_i in (1.6) possess the required properties of smoothness to ensure the existence and uniqueness of the solution of this problem. Hence it follows that the expansions in Eqs. (2.7) and (3.8) must begin with the same values of n.

After the $y_i^{(-1)}$ have been found, the values of $y_i^{(0)}, \ldots, y_i^{(n)}, \ldots$ are determined from the recurrence relations.

4. Physical Conditions of Compatibility. Relation between Stress and Displacement Wave Fields. Suppose the Laplace transform of a displacement wave is given by the series

$$\tilde{u}^j = \sum_{n=-1}^{\infty} \frac{f^{j(n)}}{p^{n+1}} e^{-p\omega}, \tag{4.1}$$

and the transform of the stress wave has the form (2.7). We substitute (4.1) and (2.7) into the transformed equation of motion (1.1), collect terms with the same powers of p, and equate them to zero. Then we obtain

$$\rho f^{j(-1)} = 0; \tag{4.2}$$

$$\rho f^{j(0)} = -z^{ij(-1)}\omega_{,i}; \tag{4.3}$$

$$\rho f^{j(n+1)} = \nabla_i z^{ij(n-1)} - z^{ij(n)}\omega_{,i} \tag{4.4}$$

for $n = 0, 1, 2, 3, \ldots$.

The condition (4.2) indicates that the order of the minimum discontinuity of the time derivative of the displacement is one lower than the minimum order of the discontinuity of the stress. By using Eqs. (4.3) and (4.4) and the series (2.6) for the known equation of the wavefront, the total field of the displacement wave can be constructed if the field of the stress wave is known. Equations (4.2)-(4.4) are valid for any continuous medium, elastic or not, since no physical laws were used in their derivation.

We now substitute (4.1) and (2.7) into the Laplace transform of Hooke's law and collect terms in the same powers of p.

Then we obtain

$$C_k^{ijl}\omega_{,l}f^{k(-1)} = 0; \tag{4.5}$$

$$z^{ij(-1)} = C_k^{ijl}\omega_{,l}f^{k(0)}; \tag{4.6}$$

$$z^{ij(n)} = C_k^{ijl}\left(\nabla_l f^{k(n)} - \omega_{,l}f^{k(n+1)}\right) \tag{4.7}$$

for $n = 0, 1, 2, 3, \ldots$. Equations (4.3) and (4.6) determine the relation between the minimum discontinuities of the stress waves and the minimum discontinuities of the displacement waves. A similar relation for discontinuities of higher derivatives is determined by Eqs. (4.4) and (4.7).

Equations (4.6) and (4.7) enable us to find the total fields of stress waves if we are given the field of the displacement wave.

Using the notation of (3.5), Eq. (4.3) can be rewritten in the form

$$z^{ij(-1)}\nu_i = -\rho G_n f^{j(0)}.$$

If $f^{j(0)} = 0$,

$$z^{ij(0)}\nu_i = -\rho G_n f^{j(1)}. \tag{4.8}$$

Equation (4.8) agrees with the condition of dynamic compatibility given in [7] and derived there in a different way.

5. Conditions of Dynamic Compactness for Stress Waves. We say that a system consisting of a semi-infinite elastic medium bounded by a surface S on which there is a distribution of active loads satisfies the condition of dynamic compactness if under the action of a boundary load for a finite time t_0 each point of the medium is at rest after the passage of all the unloading wavefronts.

It follows from this definition that dynamically compact systems transfer the given distribution of the boundary load in such a way that disturbances localized in time on the boundary correspond to disturbances localized in time at each point of space.

In accord with the definition of compactness we replace the general boundary conditions (1.6) by the conditions

$$\sigma^{ij}|_S = A^{ij}(x_\alpha)|_S P^{ij}(t) G(t, t_0) \tag{5.1}$$

(no summation over indices), where

$$G(t, t_0) = H(t) H(t_0 - t) = \begin{cases} 0, & t < 0, \\ 1, & 0 \leqslant t \leqslant t_0, \\ 0, & t > t_0. \end{cases}$$

In (5.1) the six components A^{ij} are expressed in terms of the three components of the stress vector, the gradient at the surface S, and the characteristics of the medium in the neighborhood of this surface (cf. the last three rows of matrix (3.2)). It turns out that the investigation of dynamic compactness for arbitrary functions P^{ij} reduces to veryifying this condition for

$$P^{ij}(t) = \delta(t). \tag{5.2}$$

LEMMA. A system for stresses (1.5), (5.1), (1.7), and (1.8) is dynamically compact if and only if to the boundary disturbance (5.1) for condition (5.2) there corresponds the solution

$$\sigma^{ij} = \sum_{\gamma=1}^{3} A_\gamma^{ij}(x_\alpha)\, \delta(\Omega_\gamma), \tag{5.3}$$

where $\Omega_\gamma = t - \omega_\gamma(x_\alpha) = 0$ is the equation of the corresponding front.

Proof. As noted in Sec. 3, to the boundary condition (5.1), taking account of (5.2), there must correspond, in any case, the solution

$$\sigma^{ij} = \sum_{\gamma=1}^{3} \left\{ A_\gamma^{ij}(x_\alpha)\, \delta(\Omega_\gamma) + \sum_{n=0}^{\infty} \frac{z_\gamma^{ij(n)}}{n!} \Omega_\gamma^n H(\Omega_\gamma) \right\} = \sum_{\gamma=1}^{3} \{ A_\gamma^{ij}(x_\alpha)\, \delta(\Omega_\gamma) + B_\gamma^{ij}(x_\alpha, \Omega_\gamma) H(\Omega_\gamma) \}. \tag{5.4}$$

If we consider solution (5.4) as fundamental, in accord with the principle of superposition [6], to the boundary disturbance (5.1) there must correspond the solution

$$\sigma^{ij} = \sum_{\gamma=1}^{3} \left\{ A_\gamma^{ij} P^{ij}(\Omega_\gamma) G(\Omega_\gamma, t_0) + \int_{-\infty}^{\infty} P^{ij}(\tau) G(\tau, t_0) \right.$$

$$\left. \times B_\gamma^{ij}(x_\alpha, \Omega_\gamma - \tau) H(\Omega_\gamma - \tau)\, d\tau \right\} = \sum_{\gamma=1}^{3} \left\{ A^{ij}(x_\alpha) P^{ij}(\Omega_\gamma) G(\Omega_\gamma, t_0) + \left[\int_0^{\min(\Omega_\gamma, t_0)} P^{ij}(\tau) B_\gamma^{ij}(x_\alpha, \Omega_\gamma - \tau)\, d\tau \right] H(\Omega_\gamma) \right\}$$

(i and j are not summed over).

We consider the solution at an arbitrary fixed interior point x_{α_0}. We have

$$\sigma^{ij}|_{x_{\alpha_0}} = \begin{cases} 0 & \text{for } \Omega_{\max} < 0, \\ I^{ij}(t) & \text{for } 0 \leqslant \Omega_{\max},\ \Omega_{\min} \leqslant t_0, \\ \sum_{\gamma=1}^{3} \int_0^{t_0} P^{ij}(\tau) B_\gamma^{ij}(x_{\alpha_0}, \Omega_\gamma - \tau) & \text{for } \Omega_{\min} > t_0 \end{cases} \tag{5.5}$$

(i and j are not summed over); the I^{ij} are certain functions of the time which are not generally zero. If the conditions of the lemma are satisfied, we have, from (5.4) $\sum_{\gamma=1}^{3} B_\gamma^{ij} \equiv 0$, and from (5.5) there follows (5.3), i.e., dynamic compactness. If the requirement for dynamic compactness is satisfied, it is necessary for the last expression in (5.5) to vanish for arbitrary P^{ij}. But this is equivalent to $\sum_{\gamma=1}^{3} B_{\gamma}^{ij} \equiv 0$.

THEOREM. The necessary and sufficient condition for system (1.5)-(1.8) to be dynamically compact is that the following system of equations be satisfied on all fronts simultaneously:

$$D_{lm}(z^{ij}) = M_{lm}(z^{ij}) = L_{lm}(z^{ij}) = 0, \tag{5.6}$$

where the operators D_{lm}, M_{lm}, and L_{lm} are described in (2.10)-(2.12).

Proof. We use the lemma proved. Suppose a load (5.1) acts with condition (5.2). This corresponds to the fact that the expansion of the transform of (5.1) contains just one term of series (2.7). We solve the recurrent system (2.9). After determining the $z^{ij(-1)}$, finding the $z^{ij(n)}$ (n = 0, 1, . . .) is reduced as a consequence of (5.6) to the solution of a Cauchy problem for first-order differential equations with zero initial values on S; these solutions are zero. Thus, the conditions of the lemma are satisfied. This theorem enables us in principle to isolate all classes of inhomogeneities for a given surface S when for any distributions of boundary loads the condition of dynamic compactness is satisfied. To do this it is sufficient to eliminate the functions z^{ij} from (5.6).

It is not physically obvious, but the dynamic compactness of a stress wave does not necessarily lead to dynamic compactness of the displacement wave. The situation when both waves are dynamically compact is actually exceptional. It follows from Eqs. (4.2)-(4.4) that for a system which is dynamically compact with respect to stress, the displacement field which corresponds to a boundary disturbance of the type (5.2) increases linearly with time. Therefore it is clear physically that for one-dimensional problems with cylindrical or spherical symmetry when normal stresses act on the boundary, dynamic compactness for stress waves cannot occur. This can be proven analytically also.

6. Conditions of Dynamic Compactness for a One-Dimensional Stress Wave of Rotation $\sigma_{r\varphi}$ for a Cylindrical Cavity in a Cylindrical Orthotropic Inhomogeneous Space. We consider a system consisting of a cylindrical cavity in a cylindrical orthotropic inhomogeneous space, and a boundary load $\sigma_{r\varphi}$ uniformly distributed over the surface of discontinuity for a time. The coordinate axes $(x_1, x_2, x_3) \rightarrow (r, \varphi, z)$ coincide with the axes of orthotropy. The equation of the surface of the cylindrical cavity is $r = r_0 = \text{const}$. Let us find the conditions of dynamic compactness for this system.

For cylindrical coordinates [1]

$$g_{11} = 1, \; g_{22} = r^2, \; g_{33} = 1, \; \Gamma_{22}^{1} = -r, \; \Gamma_{21}^{2} = \Gamma_{12}^{2} = r^{-1},$$

and the rest of the Christoffel symbols are zero; the stress $\sigma_{r\varphi}$ is the physical projection of the stress tensor σ^{12} and is given by

$$\sigma_{r\varphi} = \sigma^{12}\sqrt{g_{11}g_{22}} = \sigma^{12}r. \tag{6.1}$$

The conditions of dynamic compactness (5.6) for this system degenerate into the three equations

$$\left(C_{1212}\omega_{,1}^{2} - \rho r^2\right) z^{12} = 0; \tag{6.2}$$

$$\frac{3}{r} + 2\frac{z_{,1}^{12}}{z^{12}} + \frac{\zeta_{,1}}{\zeta} = 0 \quad (\zeta = \rho^{-1}\omega_{,1}); \tag{6.3}$$

$$\left[\rho^{-1}\left(z_{,1}^{12} + 3r^{-1}z^{12}\right)\right]_{,1} = 0, \tag{6.4}$$

and the remaining equations of (5.6) are satisfied identically. Here z^{12} means a discontinuity of σ^{12} of any order.

It follows from (6.2) that

$$\omega_{,1}^{2} = \rho^2\theta^{-2}\left(\theta = \sqrt{\rho C_{1212}^{*}}\right),$$

where C_{1212}^{*} is the physical projection of the stiffness tensor.

From (6.3) and (6.1) we obtain the law of variation of the stress wave amplitude at the front

$$\sigma_{r\varphi} = \sigma_0 (r^{-1}\theta)^{1/2},$$

where σ_0 = const and is determined from the magnitude of the stress at the boundary and the boundary characteristics of the medium. We obtain from (6.4) a first-order differential relation relating the density and stiffness functions,

$$\theta^{1/2} = C_1 r^{3/2} \rho \, [3r^{-1} + (\ln \theta)_{,1}]^{-1}, \qquad (6.5)$$

where C_1 is an arbitrary constant.

In the special case $\rho C^*_{1212} = C_2$ = const we obtain from (6.5)

$$\rho = C_3 r^{-1/2}; \; C^*_{1212} = C_2 C_3^{-1} r^{1/2},$$

where C_3 is an arbitrary positive constant.

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